Potential modeling with uncertain covariables

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18th Annual Conference IAMG2017
Fremantle, Australia, Sep 2-9, 2017

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The event must be sufficiently well understood in terms of cause and effect to collect data corresponding to spatially referenced covariables $B_\ell, \ell = 0, \ldots, m$, in favor or against the event T to occur – **causality modeling**.

Potential modeling with uncertain covariables

Objective of potential modeling

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Then spatially referenced "posterior" probabilities given the covariables can be estimated by several approaches including weights-of-evidence, logistic regression, compositional regression, artificial neural nets, and many others;
generally by **statistical learning** or **machine learning**.

Potential modeling with uncertain covariables

Deficiencies of Raster-Based Models

- **training region** must provide perfect knowledge (*no uncertain covariables, no missed occurrence*)
- **spatial resolution** a user decision: different resolutions may result in inconsistent models, cf. Baddeley et al., 2010
- **non-spatial** models, neglecting spatially induced dependences (*shuffle pixels, no change in results*)
- **Weights-of-Evidence** (WofE) require joint **conditional independence**: assumption cannot be relaxed while preserving method simplicity
- **logistic regression**: canonical generalization of WofE; a proper regression model may be **difficult to design**.
- **artificial neural nets** (ANN) focus on prediction, **do not provide insight and understanding**.

Potential modeling with uncertain covariables

Towards Poisson and Cox point processes

- Logistic regression with respect to (very) small pixels is approximately a Poisson point process.
- A **Poisson point process with a log-linear model of the intensity field is a canonical generalization of logistic regression: Cox process**

$$f(\boldsymbol{x};\lambda) = \exp\left(-\int_W (\lambda(u)-1)\,du\right)\prod_{i=1}^{n(\boldsymbol{x})}\lambda(x_i),$$

$$\ln\lambda(u) = B(u) + \sum_{\ell}\theta_\ell\, Z_\ell(u)$$

Baddeley, Rubak, Turner, 2015, Spatial Point Patterns: Methodology and Applications with R: Chapman and Hall/CRC
Addrian Baddeley, IAMG Matheron Lecturer 2008

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Cox point process

a Poisson process with locally-varying **intensity field** $\lambda(u)$

- total number $\mathrm{N}(\mathbf{X} \cap B)$ of random points inside a domain $B \subset \mathbb{R}^n \sim$ Poisson with parameter $\lambda(B)$

$$\mathbb{P}\Big(\mathrm{N}\left(\mathbf{X} \cap B\right) = n(\boldsymbol{x})\Big) = \exp\left(-\lambda\left(B\right)\right) \frac{\Big(\lambda\left(B\right)\Big)^{n(\boldsymbol{x})}}{n(\boldsymbol{x})!}$$

$$\mathbb{E}\Big(\mathrm{N}\left(\mathbf{X} \cap B\right)\Big) = \mathrm{Var}\Big(\mathrm{N}\left(\mathbf{X} \cap B\right)\Big) = \lambda(B) = \int_B \lambda(u) du,$$

- The Cox process with intensity field $\lambda(u) > 0$ has **density**

$$f(\boldsymbol{x}; \lambda) = \exp\left(-\int_W \left(\lambda(u) - 1\right) du\right) \prod_{i=1}^{n(\boldsymbol{x})} \lambda(x_i),$$

where $n(\boldsymbol{x})$ denotes the total number of points in $\boldsymbol{x}$, and $|W|$ is the volume of W.

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Point processes

A **spatial covariable** is a random function $(\mathbf{Z}(u), u \in W)$ observable at every location $u \in W$.

Spatial covariables may be used to model the intensity $\lambda(u)$ as

- deviations from a "baseline" $B(u)$

$$\lambda(u) = \rho(Z(u))\, B(u),$$

  with some function ρ to be estimated; or

- in terms of a **log-linear model**

$$\ln \lambda(u) = B(u) + \sum_{\ell} \theta_\ell\, Z_\ell(u)$$

If little is known about $\mathbf{Z}(u)$, modeling assumptions concerning its distribution or geostatistics may apply.

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Likelihood of a general inhomogeneous Poisson point process

For a **general inhomogeneous Poisson point process** governed by a parameter $\boldsymbol{\theta}$ given a point pattern $\boldsymbol{x}$ inside a window W the likelihood function is

$$L(\boldsymbol{\theta}|\boldsymbol{x}) = \exp\left(-\int_W (\lambda_{\boldsymbol{\theta}}(u) - 1)\, du\right) \prod_{i=1}^{n_W(\boldsymbol{x})} \lambda_{\boldsymbol{\theta}}(x_i).$$

Maximization of the log-likelihood

$$\ln L(\boldsymbol{\theta}|\boldsymbol{x}) = \sum_{i=1}^{n_W(\boldsymbol{x})} \ln \lambda_{\boldsymbol{\theta}}(x_i) - \int_W \lambda_{\boldsymbol{\theta}}(u) du$$

with classical techniques is **numerically unfeasible** in practice (too many variables, largely unbalanced datasets).

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Likelihood of log-linear Poisson point process

Log-linear model of λ

$$\lambda_{\boldsymbol{\theta}}(u) = \exp\left(\sum_{\ell=0}^{m} \theta_\ell Z_\ell(u)\right) = \exp\Big(g_{\boldsymbol{\theta}}(\boldsymbol{Z}, u)\Big)$$

leads to the **likelihood function**

$$L(\boldsymbol{\theta}|\boldsymbol{x}) = \exp\left(-\int_W \exp\Big(g_{\boldsymbol{\theta}}(\boldsymbol{Z}, u)\Big) du\right) \prod_{i=1}^{n_W(\boldsymbol{x})} \exp\Big(g_{\boldsymbol{\theta}}(\boldsymbol{Z}, x_i)\Big).$$

Potential modeling with uncertain covariables

Likelihood of log-linear Poisson point process

Log-linear model of λ

$$\lambda_{\boldsymbol{\theta}}(u) = \exp\left(\sum_{\ell=0}^{m} \theta_\ell Z_\ell(u)\right) = \exp\left(g_{\boldsymbol{\theta}}(\boldsymbol{Z}, u)\right)$$

leads to the **log-likelihood function**

$$\begin{aligned}\ln L(\boldsymbol{\theta}|\boldsymbol{x}) &= \sum_{i=1}^{n_W(\boldsymbol{x})} g_{\boldsymbol{\theta}}(\boldsymbol{Z}, x_i) - \int_W \exp\left(g_{\boldsymbol{\theta}}(\boldsymbol{Z}, u)\right) du \\ &= \sum_{\ell=0}^{m} \theta_\ell \sum_{i=1}^{n_W(\boldsymbol{x})} Z_\ell(x_i) - \int_W \exp\left(\sum_{\ell=0}^{m} \theta_\ell Z_\ell(u)\right) du.\end{aligned}$$

Under mild conditions, the MLE exists and is unique (Baddeley et al., 2015, p. 344).

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Likelihood of log-linear Poisson point process

Numerically, the MLE is the solution of the **score equations** $\boldsymbol{U}(\boldsymbol{\theta}|\boldsymbol{x}) = 0$, where the score function is

$$\boldsymbol{U}(\boldsymbol{\theta}|\boldsymbol{x}) = \sum_{i=1}^{n_W(\boldsymbol{x})} \boldsymbol{Z}(x_i) - \int_W \boldsymbol{Z}(u)\lambda_{\boldsymbol{\theta}}(u)du.$$

The score is a vector with components

$$U_\ell(\boldsymbol{\theta}|\boldsymbol{x}) = \sum_{i=1}^{n_W(\boldsymbol{x})} Z_\ell(x_i) - \int_W Z_\ell(u)\lambda_{\boldsymbol{\theta}}(u)du, \;\; \ell = 0, \ldots, m.$$

Potential modeling with uncertain covariables

Likelihood of log-linear Poisson point process

The log-linear model of λ initially presumes that $Z_\ell, \ell = 0, \ldots, m$, are known **for all** $u \in W \subset \mathbb{R}^n$.

Potential modeling with uncertain covariables

Likelihood of log-linear Poisson point process

The log-linear model of λ initially presumes that $Z_\ell, \ell = 0, \ldots, m$, are known **for all** $u \in W \subset \mathbb{R}^n$.

If they are known for a **finite set** $s_i \in \mathbb{R}^n, i = 1, \ldots, q$, of locations only, then the expectation of the likelihood function with respect to the probability of the random function $(\mathbf{Z}(s), s \in W)$ given the data $\mathbf{Z}(s_i), i = 1, \ldots, q$, applies, i.e.,

$$\mathbb{E}_{\mathbb{P}(\circ|\mathrm{data})}\Big(L(\boldsymbol{\theta}|\boldsymbol{x})\Big) =$$

$$\mathbb{E}_{\mathbb{P}(\circ|\mathrm{data})}\Big(\exp\Big(-\int_W \exp\Big(g_{\boldsymbol{\theta}}(\boldsymbol{Z}, u)\Big)du\Big) \prod_{i=1}^{n_W(\boldsymbol{x})} \exp\Big(g_{\boldsymbol{\theta}}(\boldsymbol{Z}, x_i)\Big)\Big)$$

which appears as of today **numerically intractable**.

Potential modeling with uncertain covariables

Likelihood of log-linear Poisson point process

Log-linear model of λ, i.e., log-linear random function,

$$\ln\big(\lambda_{\boldsymbol{\theta}}(u)\big) = \boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}(u)$$

leads to the likelihood function

$$L(\boldsymbol{\theta}|\boldsymbol{x}) = \exp\left(-\int_W \exp\left(\boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}(u)\right) du\right) \prod_{i=1}^{n_W(\boldsymbol{x})} \exp\left(\boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}(x_i)\right).$$

Potential modeling with uncertain covariables

Likelihood of log-linear Poisson point process

Log-linear model of λ, i.e., log-linear random function,

$$\ln\big(\lambda_{\boldsymbol{\theta}}(u)\big) = \boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}(u)$$

leads to the log-likelihood function

$$\ln L(\boldsymbol{\theta}|\boldsymbol{x}) = -\int_W \exp\big(\boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}(u)\big)du \; + \; \sum_{i=1}^{n_W(\boldsymbol{x})} \boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}(x_i).$$

Potential modeling with uncertain covariables

Likelihood of log-linear Poisson point process

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leads to the log-likelihood function

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Assuming that $\boldsymbol{Z}$ is a **Gaussian random function** with expectation $\boldsymbol{\mu}(u)$ and variogram Γ, taking the conditional expectation, i.e., kriging, leads to

$$\ln L(\boldsymbol{\theta}|\boldsymbol{x}) = -\int_W \exp\left(\boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}^*(u) + \frac{1}{2}\boldsymbol{\theta}^{\mathsf{T}} \Sigma_{\text{Krige}}\, \boldsymbol{\theta}\right) du + \sum_{i=1}^{n_W(\boldsymbol{x})} \mathbb{E}\big(\boldsymbol{\theta}^{\mathsf{T}} \boldsymbol{Z}(x_i)\big).$$

Potential modeling with uncertain covariables

Non-spatial analogue:
Poisson regression with error-prone covariables

For given $\beta_0, \beta_1, \beta_2$ and simulated normal distributed Z_1, Z_2

$$\ln \lambda = \beta_0 + \beta_1 Z_1 + \beta_2 Z_2$$

a random variable $X \sim \text{Po}(\lambda)$ is simulated.
Adding some noise $\varepsilon \sim N(0, \sigma)$

1. apply ML to estimate $\beta_0, \beta_1, \beta_2$ using X, Z_1', Z_2' with $Z_1' = Z_1 + \varepsilon, Z_2' = Z_2 + \varepsilon$,
2. apply MC simulation of Z_1', Z_2' to estimate $\beta_0, \beta_1, \beta_2$, introducing additional error,

and compare.

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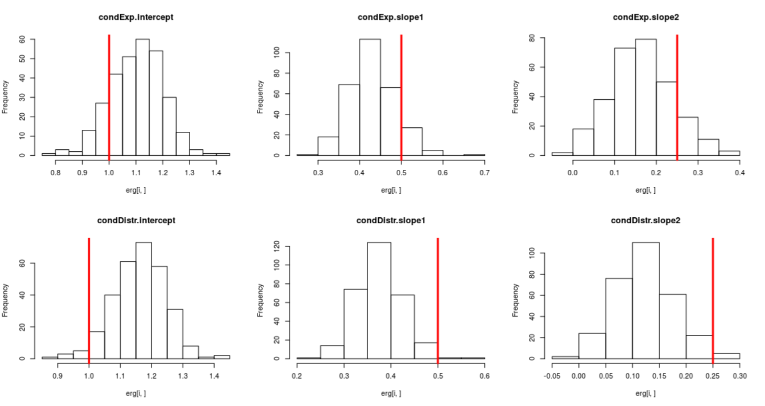

Estimation of regression parameters $\boldsymbol{\theta}$ of non-spatial Poisson regression model $\lambda_{\boldsymbol{\theta}} = \mathbb{E}(Y|\boldsymbol{Z}) = \exp(\theta_0 + \theta_1 Z_1 + \theta_2 Z_2)$ of Poisson distributed Y and Gaussian covariables Z_1, Z_2 using either "kriging" (top row) or "simulation" (bottom row).

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Conclusions

- Raster-based methods suffer from
  - requiring complete knowledge within training region,
  - neglecting spatially imposed dependence,
  - lacking consistency with respect to spatial resolution.
- Logistic regression with smaller resolution approximates Cox process, i.e.,
- Cox process is a canonical generalization of logistic regression.
- Including uncertainty in spatial predictors appears lethal.